

# Band depths based on multiple time instances

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**Abstract** Bands of vector-valued functions  $f : T \mapsto \mathbb{R}^d$  are defined by considering convex hulls generated by their values concatenated at  $m$  different values of the argument. The obtained  $m$ -bands are families of functions, ranging from the conventional band in case the time points are individually considered (for  $m = 1$ ) to the convex hull in the functional space if the number  $m$  of simultaneously considered time points becomes large enough to fill the whole time domain. These bands give rise to a depth concept that is new both for real-valued and vector-valued functions.

## 1 Introduction

The statistical concept of *depth* is well known for random vectors in the Euclidean space. It describes the relative position of  $x$  from  $\mathbb{R}^d$  with respect to a probability distribution on  $\mathbb{R}^d$  or with respect to a sample  $x_1, \dots, x_n \in \mathbb{R}^d$  from it. Given a centrally symmetric distribution (for an appropriate notion of symmetry), the point of central symmetry is the *deepest* point (center of the distribution), while the depth of outward points is low. The concept of depth has been used in the context of trimming multivariate data, to derive depth-based estimators (e.g. depth-weighted  $L$ -estimators or ranks based on the center-outward ordering induced by the depth), to assess robustness of statistical procedures, and for classification purposes, to name a few areas, see [2, 9, 17] for extensive surveys and further references.

Often, the relative position of a point  $x$  with respect to a sample is defined with respect to the convex hull of the sample or a part of the sample. For

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instance, the classical concept of the simplicial depth appears as the fraction of  $(d + 1)$ -tuples of sampled points whose convex hull contains  $x$ , see [10]. Its population version is given by the probability that  $x$  is contained in the convex hull of  $(d + 1)$  i.i.d. copies of the random vector.

In high-dimensional spaces the curse of dimensionality comes into play and the convex hull of a finite set of sampled points forms a rather “thin” set and so it is very unlikely to expect that many points belong to it. Even the convex hull of the whole sample becomes rather small if the space dimension  $d$  is much larger than the sample size  $n$ . The situation is even worse for infinite-dimensional spaces that are typical in functional data analysis. In view of this, a direct generalisation of the simplicial depth and convex hull depth concepts leads to the situation where most points in the space have depth zero, see also [8], who discuss problems inherent with the half-space depth in infinite-dimensional spaces, most importantly zero depth and the lack of consistency, see also [15].

One possible way to overcome such difficulties is to consider the depth for the collection of function values at any given time argument value  $t$  and then integrate (maybe weightedly) over the argument space. This idea goes back to [6] and has been further studied in [4, 14].

Another approach is based on considering the position of a function relative to the band generated by functions from the sample. The band generated by real-valued functions is defined as the interval-valued function determined by the pointwise minimum and maximum of the functions from the sample. The corresponding band depth has been studied in [11, 12]. In the multivariate case the band becomes a set-valued function that at each point equals the convex hull of the values of functions from the sample, see [13]. Another multivariate generalisation of the band depth in [7] is based on taking convex combinations of band depths associated to each component. Yet another multivariate functional depth concept was studied in [4] by integrating the half-space depth over the time domain, see also [3]. It is argued in [7] that the multivariate setting makes it possible to incorporate other functional data parameters, such as derivatives, into the sample. It is also possible to combine a function with its smoothed version, possibly with different bandwidths.

In this paper we suggest a new concept of multivariate functional depth based on taking convex hulls of the functions’ values at  $m \geq 1$  time points combined to build a new higher-dimensional vector. In a sense, this concept pulls together values of the function at different points and so naturally incorporates the time dependency effects, and so better reflects the shape of curves. Two examples at which these  $m$ -band depths are used are presented.

The constructions described in Section 3 remind very much the conventional simplicial band depth, where the main point is to check if a point belongs to the convex hull of a subsample. The underlying convex hull in the functional space is replaced by the band, as in [11]. It is shown that the introduced band depth satisfies the main properties described in [4, 13]. The theoretical computation of the  $m$ -band depth is usually unfeasible, since it

requires computing the probabilities that a point belongs to a convex hull of random points. Still, its empirical variant is consistent and rather easy to compute.

## 2 Regions formed by samples in functional spaces

### *m*-bands

Let  $\mathbb{E}$  be a linear space of functions  $f : T \mapsto \mathbb{R}^d$  whose argument  $t$  belongs to a rather general topological space  $T$ . For example,  $\mathbb{E}$  may be the family of continuous functions on an interval  $T$  or a collection of  $d$ -vectors if  $T$  is a finite set.

Consider functions  $f_1, \dots, f_j \in \mathbb{E}$ . The convex hull  $\text{conv}(f_1, \dots, f_j)$  of these functions is the family of functions  $f \in \mathbb{E}$  that can be represented as

$$f(t) = \sum_{i=1}^j \lambda_i f_i(t), \quad t \in T,$$

for some non-negative constants  $\lambda_1, \dots, \lambda_j$  that sum up to one.

If the coefficients  $\lambda_1, \dots, \lambda_j$  are allowed to be arbitrary functions of  $t$ , we arrive at the family of functions  $f \in \mathbb{E}$  such that, for all  $t \in T$ , the value  $f(t)$  belongs to the convex hull of  $f_1(t), \dots, f_j(t)$ . Following [11, 13] for univariate (resp. multivariate) functions, the set of such functions is called the *band* generated by  $f_1, \dots, f_j$  and is denoted by  $\text{band}(f_1, \dots, f_j)$ . It is obvious that

$$\text{conv}(f_1, \dots, f_j) \subset \text{band}(f_1, \dots, f_j).$$

If  $d = 1$  (as in [11]), then  $\text{band}(f_1, \dots, f_j)$  consists of all functions  $f$  such that

$$\min_{i=1, \dots, j} f_i(t) \leq f(t) \leq \max_{i=1, \dots, j} f_i(t), \quad t \in T. \quad (1)$$

In order to obtain a set of functions with interior points, one should avoid the case when the convex hull of  $f_1(t), \dots, f_j(t)$  is of a lower dimension than  $d$  at some  $t$ . In particular, for this  $j$  should be greater than  $d$ .

We define nested families of functions that lie between the band and the convex hull generated by the sample.

**Definition 1.** The *m*-band,  $\text{band}_m(f_1, \dots, f_j)$ , generated by  $f_1, \dots, f_j \in \mathbb{E}$  is the family of functions  $f \in \mathbb{E}$  such that, for all  $t_1, \dots, t_m \in T$ , the vector  $(f(t_1), \dots, f(t_m))$  belongs to the convex hull of  $\{(f_i(t_1), \dots, f_i(t_m)), i = 1, \dots, j\}$ , i.e.

$$(f(t_1), \dots, f(t_m)) = \sum_{i=1}^j \lambda_i (f_i(t_1), \dots, f_i(t_m)) \quad (2)$$

for non-negative real numbers  $\lambda_1, \dots, \lambda_j$  that sum up to one and may depend on  $(t_1, \dots, t_m)$ .

*Example 1 (Special cases).* If  $T = \{t\}$  is a singleton, the functions become vectors in  $\mathbb{R}^d$  and the  $m$ -band is their convex hull for all  $m \geq 1$ .

If  $T$  is a finite set of cardinality  $k$  and  $d = 1$ , then the functions  $f_1, \dots, f_j$  of  $t \in T$  can be viewed as vectors  $x_i = (x_{i1}, \dots, x_{ik}) \in \mathbb{R}^k$ ,  $i = 1, \dots, j$ . The 1-band is the smallest hyperrectangle that contains  $x_1, \dots, x_j$ , which is given by  $\times [a_l, b_l]$  for  $a_l = \min(x_{il}, i = 1, \dots, j)$  and  $b_l = \max(x_{il}, i = 1, \dots, j)$  for  $l = 1, \dots, k$ . The 2-band is obtained as the largest set such that its projections on each 2-dimensional coordinate plane equals the projection of the convex hull of  $x_1, \dots, x_k$ . The  $k$ -band coincides with the convex hull of  $x_1, \dots, x_j$ .

If  $m = 1$  and  $d = 1$ , then we recover the band introduced in [11] and given by (1), so that  $\text{band}(f_1, \dots, f_j) = \text{band}_1(f_1, \dots, f_j)$ .

If  $f \in \text{band}_m(f_1, \dots, f_j)$ , then each convex combination of the values for  $f_1, \dots, f_j$  and  $f$  can be written as a convex combination of the values of  $f_1, \dots, f_j$  and so

$$\text{band}_m(f_1, \dots, f_j) = \text{band}_m(f_1, \dots, f_j, f).$$

The  $m$ -band is additive with respect to the Minkowski (elementwise) addition. In particular,

$$\text{band}_m(g + f_1, \dots, g + f_j) = g + \text{band}_m(f_1, \dots, f_j) \quad (3)$$

for all  $g \in \mathbb{E}$ . The  $m$ -band is equivariant with respect to linear transformations, that is,

$$\text{band}_m(Af_1, \dots, Af_j) = \{Af : f \in \text{band}_m(f_1, \dots, f_j)\} \quad (4)$$

for all  $A : T \mapsto \mathbb{R}^{d \times d}$  with  $A(t)$  nonsingular for all  $t \in T$ . If all functions generating an  $m$ -band are affected by the same phase variation, the phase of the  $m$ -band is affected as shown below,

$$\text{band}_m(f_1 \circ h, \dots, f_j \circ h) = \{f \circ h : f \in \text{band}_m(f_1, \dots, f_j)\} \quad (5)$$

for any bijection  $h : T \mapsto T$ . If  $d = 1$  and  $\mathbb{E}$  consists of continuously differentiable functions on  $T = \mathbb{R}$ , then  $f \in \text{band}_m(f_1, \dots, f_j)$  yields that  $f'$  belongs to  $\text{band}_{m-1}(f'_1, \dots, f'_j)$ . This can be extended for higher derivatives.

It is obvious that  $\text{band}_m(f_1, \dots, f_j)$  is a convex subset of  $\mathbb{E}$ ; since the points  $t_1, \dots, t_m$  in Definition 1 are not necessarily distinct, it decreases if  $m$  grows. The following result shows that the  $m$ -band turns into the convex hull for large  $m$ .

**Proposition 1.** *Assume that all functions from  $\mathbb{E}$  are jointly separable, that is there exists a countable set  $Q \subset T$  such that, for all  $f \in \mathbb{E}$  and  $t \in T$ ,  $f(t)$  is the limit of  $f(t_n)$  for  $t_n \in Q$  and  $t_n \rightarrow t$ . Then, for each  $f_1, \dots, f_j \in \mathbb{E}$ ,*

$$\text{band}_m(f_1, \dots, f_j) \downarrow \text{conv}(f_1, \dots, f_j) \quad \text{as } m \rightarrow \infty.$$

*Proof.* Consider an increasing family  $T_n$  of finite subsets of  $T$  such that  $T_n \uparrow Q$  and a certain function  $f \in \mathbb{E}$ . If  $m_n$  is the cardinality of  $T_n$ , and  $f$  belongs to the  $m_n$ -band of  $f_1, \dots, f_j$ , then the values  $(f(t), t \in T_n)$  equal a convex combination of  $(f_i(t), t \in T_n)$ ,  $i = 1, \dots, j$ , with coefficients  $\lambda_{ni}$ . By passing to a subsequence, assume that  $\lambda_{ni} \rightarrow \lambda_i$  as  $n \rightarrow \infty$  for all  $i = 1, \dots, j$ . Using the nesting property of  $T_n$ , we obtain that

$$f(t) = \sum \lambda_i f_i(t), \quad t \in T_n.$$

Now it suffices to let  $n \rightarrow \infty$  and appeal to the separability of  $f$ .

Moreover, under a rather weak assumption, the  $m$ -band coincides with the convex hull for sufficiently large  $m$ . A set of points in the  $d$ -dimensional Euclidean space is said to be in general position if no  $(d - 1)$ -dimensional hyperplane contains more than  $d$  points. In particular, if the set contains at most  $d + 1$  points, they will be in general position if and only if they are all extreme points of their convex hull, equivalently, any point from their convex hull is obtained as their unique convex combination.

**Proposition 2.** *If  $j \leq d(m - 1) + 1$  and there exists  $t_1, \dots, t_{m-1} \in T$  such that the vectors  $(f_i(t_1), \dots, f_i(t_{m-1})) \in \mathbb{R}^{d(m-1)}$ ,  $i = 1, \dots, j$ , are in general position, then*

$$\text{band}_m(f_1, \dots, f_j) = \text{conv}(f_1, \dots, f_j).$$

*Proof.* Let  $f \in \text{band}_m(f_1, \dots, f_j)$ . In view of (2),  $(f(t_1), \dots, f(t_{m-1}))$  equals a convex combination of  $(f_i(t_1), \dots, f_i(t_{m-1}))$ ,  $i = 1, \dots, j$ , which is unique by the general position condition. By considering an arbitrary  $t_m \in T$ , we see that  $f(t_m)$  is obtained by the same convex combination, so that  $f$  is a convex combination of functions  $f_1, \dots, f_j$ .

In particular, if  $d = 1$ , then the 2-band of two functions coincides with their convex hull. It suffices to note that if  $f_1$  and  $f_2$  are not equal, then  $f_1(t_1)$  and  $f_2(t_1)$  are different for some  $t_1$  and so are in general position. The same holds for any dimension  $d \geq 2$ .

*Example 2 (Linear and affine functions).* Let  $f_1, \dots, f_j$  be constant functions. Then their 1-band is the collection of functions lying between the maximum and minimum values of  $f_1, \dots, f_j$ . The 2-band consists of constant functions only and coincides with the convex hull. Together with (3), this implies that the 2-band generated by functions  $f_i(t) = a(t) + b_i$ ,  $i = 1, \dots, j$ , is the set of functions  $a(t) + b$  for  $b$  from the convex hull of  $b_1, \dots, b_j$ .

If  $f_i(t) = a_i t + b_i$ ,  $i = 1, \dots, j$ , are affine functions of  $t \in \mathbb{R}$ , then their 3-band consists of affine functions only and also equals the convex hull. Indeed,

$$(f(t_1), f(t_2), f(t_3)) = \sum \lambda_i (a_i(t_1, t_2, t_3) + b_i(1, 1, 1))$$

yields that

$$\frac{f(t_3) - f(t_1)}{f(t_2) - f(t_1)} = \frac{t_3 - t_1}{t_2 - t_1}.$$

Therefore each  $f$  from  $\text{band}_3(f_1, \dots, f_j)$  is an affine function.

*Example 3 (Monotone functions).* Let  $d = 1$  and let  $f_1, \dots, f_j$  be non-decreasing (respectively non-increasing) functions. Then their 2-band is a collection of non-decreasing (resp. non-increasing) functions. If all functions  $f_1, \dots, f_j$  are convex (resp. concave), then their 3-band is a collection of convex (resp. concave) functions.

*Remark 1.* The definition of the  $m$ -band can be easily extended for subsets  $F$  of a general topological linear space  $\mathbb{E}$ . Consider a certain family of continuous linear functionals  $u_t, t \in T$ . An element  $x \in \mathbb{E}$  is said to belong to the  $m$ -band of  $F$  if for each  $t_1, \dots, t_m \in T$ , the vector  $(u_{t_1}(x), \dots, u_{t_m}(x))$  belongs to the convex hull of  $\{(u_{t_1}(y), \dots, u_{t_m}(y)) : y \in F\}$ . Then Definition 1 corresponds to the case of  $\mathbb{E}$  being a functional space and  $u_t(f) = f(t)$  for  $t \in T$ .

While the conventional closed convex hull arises as the intersection of all closed half-spaces that contain a given set, its  $m$ -band variant arises from the intersection of half spaces determined by the chosen functionals  $u_t$  for  $t \in T$ .

### Space reduction and time share

The  $m$ -band reduces to a 1-band by defining functions on the product space  $T^m$ .

**Proposition 3.** *For each  $j$ , the  $m$ -band  $\text{band}_m(f_1, \dots, f_j)$  coincides with  $\text{band}(f_1^{(m)}, \dots, f_j^{(m)})$ , where  $f_i^{(m)} : T^m \mapsto (\mathbb{R}^d)^m$  is defined as*

$$f^{(m)}(t_1, \dots, t_m) = (f(t_1), \dots, f(t_m)).$$

*Proof.* It suffices to note that  $f^{(m)}(t_1, \dots, t_m)$  belongs to the convex hull of  $f_i^{(m)}(t_1, \dots, t_m)$ ,  $i = 1, \dots, j$ , if and only if  $(f(t_1), \dots, f(t_m))$  belongs to the convex hull of  $(f_i(t_1), \dots, f_i(t_m))$ ,  $i = 1, \dots, j$ .

In the framework of Proposition 3, it is possible to introduce further bands (called *space-reduced*) by restricting the functions  $f_i^{(m)}$  to a subset  $S$  of  $T^m$ . For instance, the 1-band generated by functions  $f_1^{(2)}, \dots, f_j^{(2)}$  for the arguments  $(t_1, t_2) \in \mathbb{R}^2$  such that  $|t_1 - t_2| = h$  describes the joint behaviour of the values of functions separated by the lag  $h$ . If  $m = 1$ , then the space reduction is equivalent to restricting the parameter space, which can be useful, e.g. for discretisation purposes.

It is possible to quantify the closedness of  $f$  to the band by determining the proportion of the  $m$ -tuple of time values from  $T^m$  when the values of  $f$  belong to the band. Define the  $m$ -band time-share as

$$\begin{aligned} \text{TS}_m(f; f_1, \dots, f_j) \\ = \{(t_1, \dots, t_m) \in T^m : f^{(m)}(t_1, \dots, t_m) \in \text{conv}(\{f_i^{(m)}(t_1, \dots, t_m)\}_{i=1}^j)\}. \end{aligned}$$

If the functions take values in  $\mathbb{R}$ , then  $\text{TS}_1(f; f_1, \dots, f_j)$  turns into the modified band depth defined in [11, Sec. 5]. If  $f$  belongs to the  $m$ -band of  $f_1, \dots, f_j$ , then  $\text{TS}_m(f; f_1, \dots, f_j) = T^m$ , while if  $f$  belongs to the 1-band of  $f_1, \dots, f_j$ , then  $\{(t, \dots, t) : t \in T\} \subset \text{TS}_m(f; f_1, \dots, f_j)$ . It is also straightforward to incorporate the space reduction by replacing  $T^m$  with a subset  $S$ .

### 3 Simplicial-type band depths

#### Band depth

In the following, we consider the event that a function  $f$  belongs to a band generated by i.i.d. random functions  $\xi_1, \dots, \xi_j$  with the common distribution  $P$ . The  $m$ -band depth of the function  $f$  with respect to  $P$  is defined by

$$\begin{aligned} \text{bd}_m^{(j)}(f; P) &= \mathbf{P}\{f \in \text{band}_m(\xi_1, \dots, \xi_j)\} \\ &= \mathbf{P}\{(f(t_1), \dots, f(t_m)) \in \text{conv}(\{(\xi_i(t_1), \dots, \xi_i(t_m))\}_{i=1}^j), \forall t_1, \dots, t_m \in T\}. \end{aligned} \quad (6)$$

If  $m$  increases, then the  $m$ -band narrows, and so the  $m$ -band depth decreases.

We recall that when  $d = 1$  the 1-band coincides with the band introduced in [11]. Nevertheless the band depth defined in [11] is the sum of  $\text{bd}_m^{(j)}(f; P)$  with  $j$  ranging from 2 to a fixed value  $J$ . The same construction can be applied to our  $m$ -bands.

The  $m$ -band depth of  $f$  is influenced by the choice of  $j$ , and it increases with  $j$ . Unlike to the finite-dimensional setting, where  $j$  is typically chosen as the dimension of the space plus one [10], there is no canonical choice of  $j$  for the functional spaces. In order to ensure that the  $m$ -band generated by  $\xi_1, \dots, \xi_j$  differs from the convex hull, it is essential to choose  $j$  sufficiently large, and in any case at least  $d(m-1) + 2$ , see Proposition 2. Furthermore, we must impose stronger conditions on  $j$  to avoid the zero-depth problem.

**Proposition 4.** *If  $j \leq dm$  and the joint distribution of the marginals of  $P$  at some fixed  $m$  time points is absolutely continuous, then  $\text{bd}_m^{(j)}(\cdot; P) = 0$ .*

*Proof.* If  $j \leq dm$  and  $\{(\xi_i(t_1), \dots, \xi_i(t_m))\}_{i=1}^j$  are independent and absolutely continuous in  $\mathbb{R}^{dm}$ , the probability that any fixed  $x \in \mathbb{R}^{dm}$  lies in their convex hull is zero.

A theoretical calculation of the  $m$ -band depth given by (6) is not feasible in most cases. In applications, it can be replaced by its empirical variant defined in exactly the same way as in [11] for the 1-band case. Let  $f_1, \dots, f_n$  be a sample from  $P$ . Fix any  $j \in \{dm + 1, \dots, n\}$  and define

$$\text{bd}_m^{(j)}(f; f_1, \dots, f_n) = \binom{n}{j}^{-1} \sum_{1 \leq i_1 < \dots < i_j \leq n} \mathbf{1}_{f \in \text{band}_m(f_{i_1}, \dots, f_{i_j})},$$

so that  $\text{bd}_m^{(j)}(f; f_1, \dots, f_n)$  is the proportion of  $j$ -tuples from  $f_1, \dots, f_n$  such that  $f$  lies in the  $m$ -band generated by the  $j$ -tuple. The choice of  $j$  affects the results. It is computationally advantageous to keep  $j$  small, while it is also possible to sum up the depths over a range of the values for  $j$ , as in [11].

### Time-share depth

Assume now that  $T$  is equipped with a probability measure  $\mu$ , for example, the normalised Lebesgue measure in case  $T$  is a bounded subset of the Euclidean space or the normalised counting measure if  $T$  is discrete. Extend  $\mu$  to the product measure  $\mu^{(m)}$  on  $T^m$ . Define the time-share depth by

$$\text{td}_m^{(j)}(f; P) = \mathbf{E} \mu^{(m)}(\text{TS}_m(f; \xi_1, \dots, \xi_j)).$$

If  $T$  is a subset of the Euclidean space, Fubini's Theorem yields that the time-share depth is the average of the probability that  $(f(t_1), \dots, f(t_m))$  lies in the convex hull of  $j$  points in  $\mathbb{R}^{dm}$ ,

$$\text{td}_m^{(j)}(f; P) = \int \mathbf{P}\{(f(t_1), \dots, f(t_m)) \in \text{conv}(\{(\xi_i(t_1), \dots, \xi_i(t_m))\}_{i=1}^j)\} d\mu^{(m)}(t_1, \dots, t_m). \quad (7)$$

For any  $j \in \{dm + 1, \dots, n\}$ , the empirical time-share depth is given by

$$\text{td}_m^{(j)}(f; f_1, \dots, f_n) = \binom{n}{j}^{-1} \sum_{1 \leq i_1 < \dots < i_j \leq n} \mu^{(m)}(\text{TS}_m(f; f_{i_1}, \dots, f_{i_j})),$$

*Example 4 (Univariate case).* Assume that  $T$  is a singleton. Then necessarily  $m = 1$ , the function  $f$  is represented by a point  $x$  in  $\mathbb{R}^d$ , and the band depth of  $x$  for  $j = d + 1$  coincides with the simplicial depth, see [10].

*Example 5.* Let  $\xi(t) = a(t) + X$ ,  $t \in T$ , where  $X$  is a random variable. Then  $\text{band}(\xi_1, \dots, \xi_j)$  for i.i.d.  $\xi_i(t) = a(t) + X_i$ ,  $i = 1, \dots, j$ , is the set of functions bounded above by  $a(t) + \max X_i$  and below by  $a(t) + \min X_i$ . Then

$$\text{bd}_1^{(j)}(a; P) = 1 - \mathbf{P}\{X > 0\}^j - \mathbf{P}\{X < 0\}^j.$$



By Example 2,  $\text{band}_2(\xi_1, \dots, \xi_j)$  consists of functions  $a(t) + b$  for the constant  $b \in [\min X_i, \max X_i]$ . Only such functions may have a positive 2-band depth.

*Example 6.* Let now  $\xi(t) = a(t) + X$ , where  $a : T \rightarrow \mathbb{R}^d$  and  $X$  is an absolutely continuous random vector in  $\mathbb{R}^d$  which is angularly symmetric about the origin. Then

$$\text{bd}_1^{(j)}(a; P) = 1 - 2^{1-j} \sum_{i=0}^{d-1} \binom{j-1}{i} \quad (8)$$

being the probability that the origin belongs to the convex hull of  $X_1, \dots, X_j$ , see [16].

### Properties of the band depths

**Theorem 1.** *For any  $j \geq dm + 1$  we have:*

1. affine invariance.  $\text{bd}_m^{(j)}(Af + g; P_{A,g}) = \text{bd}_m^{(j)}(f; P)$  and  $\text{td}_m^{(j)}(Af + g; P_{A,g}) = \text{td}_m^{(j)}(f; P)$  for all  $g \in \mathbb{E}$  and  $A : T \mapsto \mathbb{R}^{d \times d}$  with  $A(t)$  nonsingular for  $t \in T$ .
2. phase invariance.  $\text{bd}_m^{(j)}(f \circ h; P^h) = \text{bd}_m^{(j)}(f; P)$  for any one-to-one transformation  $h : T \mapsto T$ , where  $P^h(F) = P(F \circ h^{-1})$  for any measurable subset  $F$  of  $\mathbb{E}$  when  $h^{-1}$  is the inverse mapping of  $h$ .
3. vanishing at infinity.  $\text{bd}_m^{(j)}(f; P) \rightarrow 0$  if the supremum of  $\|f\|$  over  $T$  converges to infinity, and  $\text{td}_m^{(j)}(f; P) \rightarrow 0$  if the infimum of  $\|f\|$  over  $T$  converges to infinity.

The affine invariance of both depths follows from the affine invariance of the  $m$ -bands, see (3), (4), while the phase-invariance of the band depth follows from (5).

In practice, the functions are going to be evaluated over a finite set of time points, thus  $T = \{t_1, \dots, t_k\}$  and probability  $P$  is a distribution on  $(\mathbb{R}^d)^k$ . Furthermore, the sample of functions  $f_1, \dots, f_n$  to be used to determine an empirical  $m$ -band depth should have size at least  $n \geq j \geq dm + 1$ .

**Theorem 2.** *If  $P$  is absolutely continuous, for any  $n \geq j \geq dm + 1$  we have:*

4. maximality at the center. if  $P$  is angularly symmetric about the point  $(f(t_1), \dots, f(t_k))$ , function  $f$  will be the deepest with regard to the time-share depth, and  $\text{td}_m^{(j)}(f; P) = 1 - 2^{1-j} \sum_{i=0}^{dm-1} \binom{j-1}{i}$ .
5. consistency. band depth  $\sup_{f \in \mathbb{E}} |\text{bd}_m^{(j)}(f; f_1, \dots, f_n) - \text{bd}_m^{(j)}(f; P)| \rightarrow 0$  a.s. and time-share depth  $\sup_{f \in \mathbb{E}} |\text{td}_m^{(j)}(f; f_1, \dots, f_n) - \text{td}_m^{(j)}(f; P)| \rightarrow 0$  a.s.

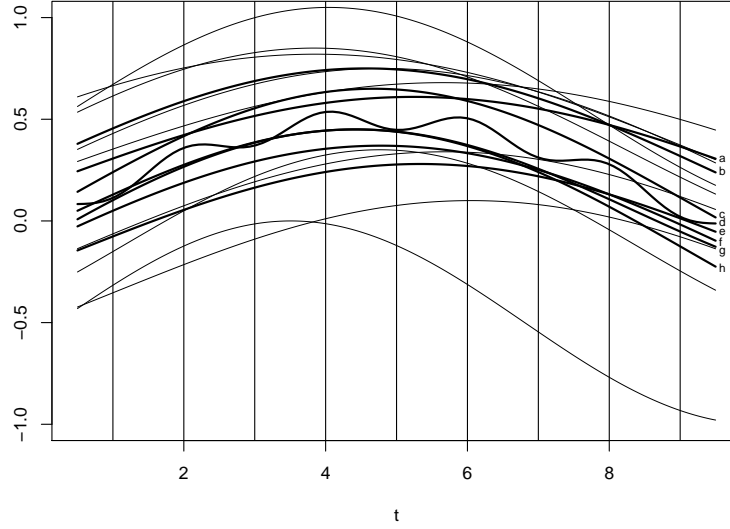
The properties of the time-share depth rely on formula (7) that makes it possible to write it as an average of the probability that a point lies in the convex hull of independent copies of a random vector. The maximality at center follows from the main result in [16] which determines the probability

inside the integral in (7), see (8), while the consistency can be proved in a similar way to [13, Th.3] extending the uniform consistency of the empirical simplicial depth [5, Th.1] to the one of the probability that a point lies in the convex hull of a fixed number of independent copies of a random vector. Such an extension, which relies on probabilities of intersections of open half-spaces, can be adapted to prove the consistency of the empirical  $m$ -band depth.

## 4 Data examples

### Simulated data

Fig. 1 shows 17 curves which are evaluated at  $T = \{1, 2, \dots, 9\}$ . Among the 17 curves, there is a clear shape outlier (marked as **d**) that lies *deep* within the bunch of curves. Such an outlier will not be detected by the outliergram from [1] due to its high depth value with regard to both of the 1-band depth and half-region depth (see [12]). Nevertheless, its anomalous shape is detected by any  $m$ -band depth with  $m \geq 2$ .

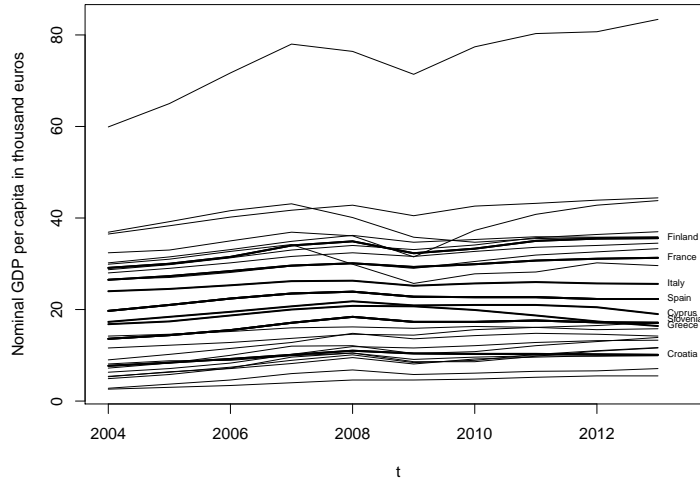


**Fig. 1** 17 curves evaluated at  $\{1, 2, \dots, 9\}$ . The eight deepest curves are thicker than the others and each of them is assigned a letter from **a** to **h**. Five deepest curves for  $\text{bd}_1^{(4)}$  (in order): **d, c, f, g, a**, for  $\text{td}_1^{(4)}$ : **d, c, g, a, f**, for  $\text{bd}_2^{(4)}$ : **g, b, f, e, a**, and for  $\text{td}_2^{(4)}$ : **g, f, c, d, h**.

It is remarkable that curve **d**, which is the deepest curve with respect to the usual band depth and modified band depth ( $\text{bd}_1^{(4)}$  and  $\text{td}_1^{(4)}$ ) is among the less deep curves for the 2-band depth ( $\text{bd}_2^{(4)}$ ) and is only the fourth deepest curve for its time-share depth ( $\text{td}_2^{(4)}$ ). The reason for this last fact is that if we restrict to either of the sets of time points  $\{1, 3, 5, 7, 9\}$  or  $\{2, 4, 6, 8\}$ , curve **d** is not a shape outlier with respect to them.

## Real data

The nominal Gross Domestic Product per capita of the 28 countries of the European Union (2004–2013) was obtained from the EUROSTAT web-site and is represented in Fig. 2. The missing observation that corresponds to Greece, 2013 was replaced by the value obtained from the FOCUSECONOMICS web-site.



**Fig. 2** Evolution of the nominal GDP per capita between 2004 and 2013 at the EU countries. Five deepest curves for  $\text{bd}_1^{(5)}$  (in order): Cyprus, Spain, Italy, Greece, and Slovenia, for  $\text{bd}_2^{(5)}$ : Spain, Slovenia, France, Croatia, and Finland, and for  $\text{bd}_2^{(5)}$  space-reduced with  $S = \{(t_1, t_2) : |t_1 - t_2| = 1\}$ : Croatia, Slovenia, Spain, Finland, and France.

The deepest curve with regard to the band depth ( $\text{bd}_1^{(5)}$ ) is the one of Cyprus. Interestingly, Cyprus suffered the 2012-13 Cypriot financial crisis at the end of the considered period and its GDP per capita experienced a decay in 2013 in comparison with its 2012 figure much greater than the one of any

other of the EU countries. Also the Greek curve is among the five deepest ones for  $\text{bd}_1^{(5)}$  despite being the only country with a constant decrement in the second half of the considered time period. If we consider 2-bands, that take into account the shape of the curves, these two curves are not any more considered representative of the evolution of the GDP per capita in the EU.

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